Aggregate data and compositional variables

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Introduction

In many applications, individual data are not widely available, if at all, especially when small geographic areas or vulnerable data are involved. It is more frequent to find aggregate data instead. Two problems arise :

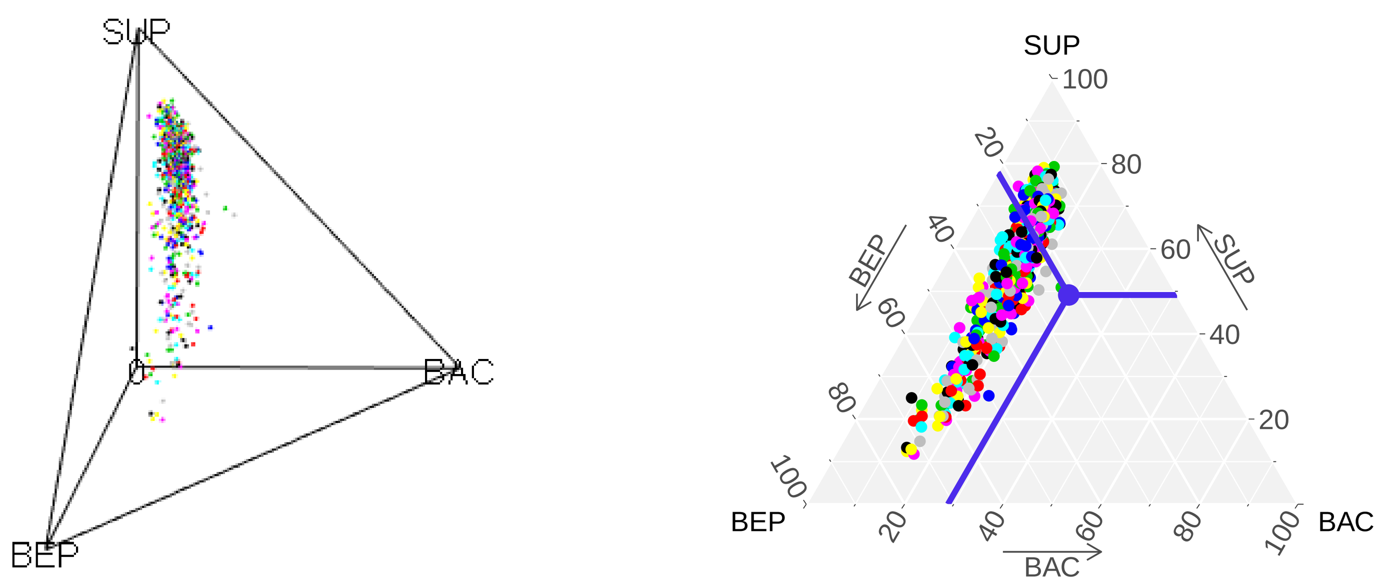
- Can these aggregate data be used to **infer** individual behavior (we will use the term of **ecological inference**) ?
- How to **manipulate** categorical variables (which become **compositional variables** once aggregated) ?

Compositional Data

A composition of D components is a vector $\mathbf{x}$ of the simplex S^D defined as

$$S^D = \left\{ \mathbf{x} = [x_1, x_2, \ldots, x_D];\ x_i > 0, i = 1, 2, \ldots, D;\ \sum_{i=1}^{D} x_i = 1 \right\} \quad (1)$$

This sample space is a $(D-1)$-dimensional subset of $\mathbb{R}^D$. For example, S^3 is a triangle (**ternary diagram**).



Picpus (12^e) : 22% BEP, 29% BAC, 79% SUP

Aitchison (1986) introduces operations on the simplex (sum and scalar multiplication). The **inner product** of two compositions is defined as :

$$\langle \mathbf{x}, \mathbf{y} \rangle_a = \sum_{i=1}^{D} \log \frac{x_i}{g(\mathbf{x})} \log \frac{y_i}{g(\mathbf{y})} = \frac{1}{D} \sum_{i<j} \log \frac{x_i}{x_j} \log \frac{y_i}{y_j} \quad (2)$$

The **additive** log-ratio transformation :

$$alr(\mathbf{x}) = [\log \frac{x_1}{x_D}, \log \frac{x_2}{x_D}, \ldots, \log \frac{x_{D-1}}{x_D}] \quad (3)$$

The **centered** log-ratio transformation :

$$clr(\mathbf{x}) = [\log \frac{x_1}{g(\mathbf{x})}, \log \frac{x_2}{g(\mathbf{x})}, \ldots, \log \frac{x_D}{g(\mathbf{x})}] \quad (4)$$

The **isometric** log-ratio transformation :

$$ilr(\mathbf{x}) = [\langle \mathbf{x}, e_1 \rangle_a, \langle \mathbf{x}, e_2 \rangle_a, \ldots, \langle \mathbf{x}, e_{D-1} \rangle_a] \quad (5)$$

Where $e_1, e_2, \ldots, e_{D-1}$ represent an orthonormal basis of S^D and $g(\mathbf{x}) = (x_1 x_2 \ldots x_D)^{1/D}$ is the geometric mean.

Econometrics with Compositional Data

Regression analysis is generally used in statistics to study the relation between a response Y and a set of explanatory covariate $\mathbf{x} = (x_1, \ldots, x_D)$ as

$$\mathbf{y} = \beta_0 + \beta^T \mathbf{x} + \epsilon \quad (6)$$

The standard linear model is unreasonable with compositional data.

- Compositions contain only **relative information** : least squares methods examine continuous variables that are linked with an absolute relationship
- When one part is altered, **another is altered** : the interpretation of the linear regressions coefficients assumes that "all other things remaining equal"
- Composition covariates involve **multicollinearity** problems
- The simplex has **particular geometric** properties and operations : most common statistical procedures are developed in the usual Euclidean geometry

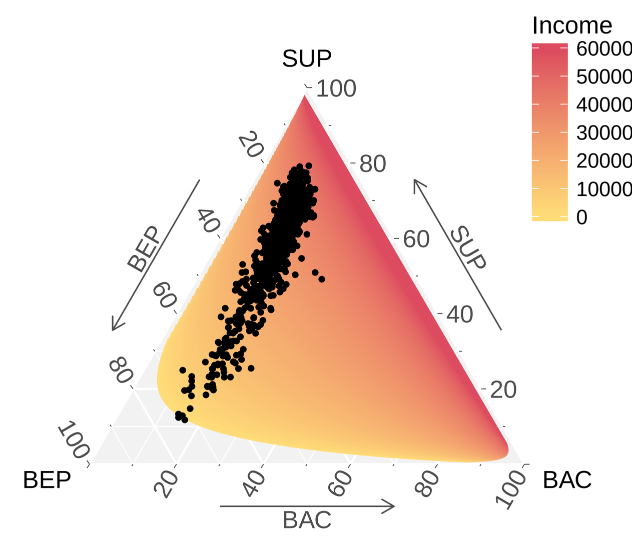
The *ilr* transformation is an isometric transformation from the simplex to the real space with usual Euclidean geometry. It allows the application of standard models to compositions as

$$\mathbf{y} = \beta_0 + \beta^T ilr(\mathbf{x}) + \epsilon \quad (7)$$

Using the estimation of β, it is possible to find b with the ilr-inverse transformation. Composition b can be interpreted as the slope parameter of the standard regression. If x differs by $\frac{b}{\|b\|}$ in the direction of b, then y differs by $\|b\|$.

An example

Analysis of median income according to diploma levels in Paris by iris.



Median income increases in the SUP direction and decreases in the BEP direction. In other words, when the proportion of SUP increases or the proportion of BEP decreases in an area, then median income increases.

Note : French diploma levels are BEP (Before A-Levels), BAC (Baccalaureate, A-Levels) and SUP (Bachelor, Master, Ph.D.)

Ecological inference

Ecological inference consists of making inference of **individual** behaviour using **aggregated** group-level data. Group-level data are generally less reliable and subject to biases and imprecision that can lead to mistake of inference. Robinson (1950) warms to be cautious when using aggregate data to study individuals ("ecological fallacy").

	Republican	Democrate	Total
Women	?	?	ϕ_j
Men	?	?	$1-\phi_j$
Total	p_j	$1-p_j$	1

The aim is to find the probability of being republican conditionally of being a man (β_j^0) or a woman (β_j^1) in the area j.

Ecological regression (Goodman, 1953) assumes that these two probabilities are **constant** over area and are estimated by least squares as

$$p_j = \beta^1 \phi_j + \beta^0 (1 - \phi_j) \quad (8)$$

The method of bounds (Duncan & Davis, 1953) is to find the minimum and maximum of the probability with

$$\max \left\{ 0; \frac{p_j - (1-\phi_j)}{\phi_j} \right\} \leq \beta_j^1 \leq \min \left\{ \frac{p_j}{\phi_j}; 1 \right\} \quad (9)$$

$$\max \left\{ 0; \frac{p_j - \phi_j}{1-\phi_j} \right\} \leq \beta_j^0 \leq \min \left\{ \frac{p_j}{1-\phi_j}; 1 \right\} \quad (10)$$

King's solution (King, 1997) is an improvement in ecological inference by combining the Goodman method and the information of the bounds to improve inference. β_j^0 and β_j^1 are linked by **tomography line** within the unit square as follows

$$\beta_j^0 = \frac{p_j}{1-\phi_j} - \frac{\phi_j}{1-\phi_j} \beta_j^1 \quad (11)$$

King suggests three assumptions :

- β_j^0 and β_j^1 are in a single cluster that is generated by a **truncated normal** bivariate distribution conditional of ϕ_j
- Absence of **spatial autocorrelation** : the number of exposed cases of one area is not related to the number of cases in the other areas
- Absence of **aggregation bias** : independence between the regressors ϕ_j and the parameters β_j^0 and β_j^1

References

Aitchison, J. (1986). *The Statistical Analysis of Compositional Data*. Chapman and Hall London.

Duncan, O., & Davis, B. (1953, dec). An Alternative to Ecological Correlation. *American Sociological Review*, *18*(6), 665-666.

Goodman, L. (1953, dec). Ecological Regressions and Behavior of Individuals. *American Sociological Review*, *18*(6), 663-664.

King, G. (1997). *A Solution to the Ecological Inference Problem*. Princeton University Press.

Robinson, W. (1950, jun). Ecological Correlation and the Behavior of Individuals. *American Sociological Review*, *15*(3), 351-357.